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ON A SUBSPACE OF A SPECIAL FINSLER SPACE

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Abstract. The present paper deals with the properties of a Finsler space F_n^* whose metric is obtained from the metric of another Finsler space F_n defined over the same manifold, with the help of a contravariant vector $v^i(x^j)$ satisfying the condition $LC_{jkr}v^r = \rho h_{jk}$, where L , h_{jk} and C_{jkr} are metric function, angular metric tensor and Cartan tensor of F_n , respectively, and ρ is a scalar function of positional coordinates x^i . Apart from obtaining expressions for different geometric objects of F_n^* , a subspace of F_n^* is studied. Apart from other results for the subspace of F_n^* , certain conditions for a subspace of F_n^* to be totally geodesic and projectively flat have been obtained.

Keywords: Finsler space; subspace; projective change; totally geodesic subspace; projectively flat space.

1. Introduction

In 1952, S. Kikuchi [11] studied the theory of a subspace of a Finsler space. H. Rund [3] in 1959, H. Yasuda [4] in 1987, T. Sakaguchi [12] in 1988 and many others mathematicians contributed significantly to the theory of Finsler subspaces and obtained many important and interesting results. In 1980 during the study of conformally flat Finsler spaces, H. Izumi [2] introduced a vector b_i which is v -covariant constant ($b_i|_j = 0$) and satisfies the condition $LC_{jk}^r b_r = \rho h_{jk}$, where ρ is a scalar independent of directional arguments y^i . He called such vector b_i as h -vector. In 1990, B. N. Prasad [1] studied a Finsler space with a special metric $ds = (g_{ij}(dx)dx^i dx^j)^{1/2} + b_i(x, y)dx^i$, where b_i is an h -vector, and obtained the Cartan connections. In 2008, M. K. Gupta and P. N. Pandey [6] worked on subspaces of a Finsler space with a special metric by taking this h -vector.

Let $F_n = (M_n, L)$ be a Finsler space and $F_n^* = (M_n, L^*)$ be another Finsler space over the same manifold M_n , whose metric L^* is obtained from the metric L of F_n by

$$(1.1) \quad L^*(x, y) = L(x, y) + v_i(x, y)y^i,$$

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where $v_i = g_{ij}v^j$, g_{ij} is the metric tensor of F_n and $v^i(x^j)$ is a contravariant vector satisfying

$$(1.2) \quad LC_{jkr}v^r = \rho h_{jk},$$

where ρ is a scalar function of positional coordinates x^i .

We call such a Finsler space $F_n^* = (M_n, L^*)$ as a special Finsler space. This special Finsler space F_n^* is a generalization of the Finsler spaces considered by the authors ([1], [6]). The aim of the present paper is to obtain the Cartan connections and to study a subspace of the Finsler space $F_n^* = (M_n, L^*)$.

2. Preliminaries

Let the Cartan connection of an n -dimensional Finsler space $F_n = (M_n, L)$ is given by the triad $CT = (F_{jk}^i, G_j^i, C_{jk}^i)$, where $G_j^i = F_{jk}^i y^k$ and C_{jk}^i is the associated Cartan tensor. If $X_i(x, y)$ be a covariant vector field then its h - and v - covariant derivatives with respect to the Cartan connection CT are given by

$$(2.1) \quad X_{i|k} = \partial_k X_i - (\dot{\partial}_r X_i) G_k^r - X_r F_{ik}^r$$

and

$$(2.2) \quad X_i|_k = \dot{\partial}_k X_i - X_r C_{ik}^r$$

respectively. Here ∂_k and $\dot{\partial}_k$ denote the partial derivatives with respect to x^k and y^k respectively, and ∂_k and $\dot{\partial}_k$ stand for $\partial/\partial x^k$ and $\partial/\partial y^k$ respectively.

The components of the metric tensor g_{ij} and the angular metric tensor h_{ij} of the Finsler space $F_n = (M_n, L)$ are defined respectively by

$$(2.3) \quad g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$$

and

$$(2.4) \quad h_{ij} = L \dot{\partial}_i \dot{\partial}_j L.$$

Differentiating (2.3) partially with respect to y^k , we obtain a tensor C_{ijk} of type $(0, 3)$ defined by

$$(2.5) \quad C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}.$$

This tensor is called the Cartan tensor and its degree of homogeneity in y^i is -1 . The normalized supporting element $l_i = y_i/L$ satisfies $l_i = \dot{\partial}_i L$. From the equations (2.3) and (2.4), we obtain the relation

$$(2.6) \quad g_{ij} = h_{ij} + l_i l_j$$

among the metric tensor g_{ij} , the angular metric tensor h_{ij} and the normalized supporting element l_i . The h -covariant derivatives and v -covariant derivatives of g_{ij} , h_{ij} and l_i satisfy [9]

$$(2.7) \quad \begin{array}{lll} \text{(a)} & g_{ij|k} = 0 & \text{(b)} \quad h_{ij|k} = 0 \quad \text{(c)} \quad L_{|i} = 0 \\ \text{(d)} & l_{|j}^i = 0 & \text{(e)} \quad l_{i|j} = \frac{1}{L} h_{ij} \quad \text{(f)} \quad L_{|i} = l_i. \end{array}$$

Let M_m ($1 < m < n$) be an m -dimensional subspace of the n -dimensional manifold M_n represented parametrically by the equations

$$(2.8) \quad x^i = x^i(u^\alpha) \quad i = 1, 2, \dots, n; \quad \alpha = 1, 2, \dots, m,$$

where u^α denote the Gaussian coordinates on the subspace M_m .

Let $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ be the projection factors [3] and the matrix $||B_\alpha^i||$ of this projection factors be supposed to be of rank m . If y^i , the supporting element, is assumed to be tangential to the subspace M_m then it can be written in terms of the projection factors as

$$(2.9) \quad y^i = B_\alpha^i(u) w^\alpha, \quad \alpha = 1, 2, \dots, m.$$

Here $w = (w^\alpha)$ is assumed to be the supporting element at the point (u^α) of the subspace M_m . The metric $L(x, y)$ of the Finsler space $F_n = (M_n, L)$ induces the metric

$$(2.10) \quad \bar{L}(u, w) = L(x(u), y(u, w))$$

on the subspace M_m . Thus, we obtain an m -dimensional Finsler subspace $F_m = (M_m, \bar{L}(u, w))$ of the space $F_n = (M_n, L)$.

Let $g_{\alpha\beta}(u, w)$ defined by

$$(2.11) \quad g_{\alpha\beta}(u, w) = \frac{1}{2} \frac{\partial^2 \bar{L}^2}{\partial w^\alpha \partial w^\beta},$$

be the metric tensor of the subspace F_m . Successive partial differentiations of (2.10) with respect to w^α and w^β give

$$(2.12) \quad g_{\alpha\beta}(u, w) = g_{ij}(x, y) B_\alpha^i B_\beta^j.$$

A covariant vector Y_i which satisfies the condition

$$(2.13) \quad Y_i B_\alpha^i(u) = 0$$

is called normal to the subspace F_m . Clearly, these are m equations for determinations of n functions Y_i . So, there exist $(n - m)$ linearly independent and mutually orthogonal unit vectors $Y_{(a)}^i$, (say), satisfying the following conditions

$$(2.14) \quad g_{ij} B_\alpha^i Y_{(a)}^j = 0$$

and

$$(2.15) \quad Y_i^{(a)} = g_{ij} Y_{(a)}^j,$$

where $(a) = m + 1, m + 2, \dots, n$. Further, (2.14) and (2.15) imply that

$$(2.16) \quad g_{ij} Y_{(a)}^i Y_{(b)}^j = \delta_{(a)(b)}, \quad \{(a), (b) = m + 1, m + 2, \dots, n\}.$$

If $B_i^\alpha(u, w)$ is the reciprocal of the projection factors B_α^i defined by

$$(2.17) \quad B_i^\alpha(u, w) = g^{\alpha\beta} B_\beta^j g_{ij},$$

then, in view of (2.12), we have

$$(2.18) \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta.$$

From (2.14), (2.15), (2.16), (2.17) and (2.18), we have

$$(2.19) \quad \begin{array}{ll} \text{(a)} & B_\alpha^i Y_i^{(a)} = 0 \\ \text{(b)} & Y_{(a)}^i B_i^\alpha = 0 \\ \text{(c)} & Y_{(a)}^i Y_i^{(b)} = \delta_{(b)}^{(a)} \\ \text{(d)} & B_\alpha^i B_j^\alpha + Y_{(a)}^i Y_j^{(a)} = \delta_j^i. \end{array}$$

If the triad $ICT = (F_{\beta\gamma}^\alpha, G_\beta^\alpha, C_{\beta\gamma}^\alpha)$, where $G_\beta^\alpha = F_{\beta\gamma}^\alpha y^\gamma$, is the induced Cartan connection of the Finsler subspace F_m then the second fundamental tensor $H_{\alpha\beta}^{(a)}$ and the normal curvature vector $H_\alpha^{(a)}$ with respect to induced Cartan connection ICT can be expressed in the direction of the normal vector $Y_{(a)}^i$ by

$$(2.20) \quad H_{\alpha\beta}^{(a)} = Y_i^{(a)} (B_{\alpha\beta}^i + F_{jk}^i B_\alpha^j B_\beta^k) + M_{(b)\alpha}^{(a)} H_\beta^{(b)}$$

and

$$(2.21) \quad H_\alpha^{(a)} = Y_i^{(a)} (B_{0\alpha}^i + F_{0j}^i B_\alpha^j)$$

respectively, where

$$(2.22) \quad M_{(b)\alpha}^{(a)} = C_{jk}^i Y_i^{(a)} Y_{(b)}^j B_\alpha^k,$$

$$(2.23) \quad B_{\alpha\beta}^i = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad B_{0\alpha}^i = v^\beta B_{\beta\alpha}^i.$$

The contraction of (2.20) by v^α gives us

$$(2.24) \quad H_{0\beta}^{(a)} = v^\alpha H_{\alpha\beta}^{(a)} = H_\beta^{(a)}.$$

3. The Finsler space $F_n^* = (M_n, L^*)$

Let $v^i = v^i(x^j)$ be a contravariant vector field in a Finsler space $F_n = (M_n, L)$ satisfying the condition (1.2).

Differentiating $v_i = g_{ir}(x, y)v^r$ partially with respect to y^j and using the condition (1.2), we obtain

$$(3.1) \quad L(\dot{\partial}_j v_i) = 2\rho h_{ij}.$$

Consider an n -dimensional Finsler space $F_n^* = (M_n, L^*)$ whose metric function $L^*(x, y)$ is obtained from the metric of the space F_n by the transformation (1.1). Throughout the paper, the geometric objects related to F_n^* will be asterisked $*$.

Differentiating (1.1) partially with respect to y^k and using (3.1), we get

$$(3.2) \quad L_k^* = L_k + v_k,$$

where $L_k^* = \dot{\partial}_k L^*$.

The normalized supporting element l_i^* of F_n^* can be written as

$$(3.3) \quad l_k^* = l_k + v_k.$$

Differentiating (3.2) partially with respect to y^j and using (3.1), we obtain

$$(3.4) \quad L_{jk}^* = L_{jk} + 2\rho h_{jk}/L,$$

where $L_{jk} = \dot{\partial}_j \dot{\partial}_k L$ and $L_{jk}^* = \dot{\partial}_j \dot{\partial}_k L^*$.

Using (2.4) in (3.4), we get

$$(3.5) \quad L_{jk}^* = (1 + 2\rho)L_{jk}.$$

Partial Differentiation of (3.5) with respect to y^i gives

$$(3.6) \quad L_{ijk}^* = (1 + 2\rho)L_{ijk},$$

where $L_{ijk} = \dot{\partial}_k L_{ij}$ and $L_{ijk}^* = \dot{\partial}_k L_{ij}^*$.

In view of (2.4), the angular metric tensor h_{ij}^* of the Finsler space F_n^* is given as

$$(3.7) \quad h_{ij}^* = \tau(1 + 2\rho)h_{ij},$$

where $\tau = \frac{L^*}{L}$.

From (3.3), (3.7) and (2.6), the fundamental metric tensor g_{ij}^* of the Finsler space F_n^* is given by

$$(3.8) \quad g_{ij}^* = \tau(1 + 2\rho)g_{ij} + v_i v_j + l_i v_j + v_i l_j + (1 - \tau(1 + 2\rho))l_i l_j.$$

Keeping $g^{*ij}g_{jk}^* = \delta_k^i$ in view, the inverse metric tensor g^{*ij} is given by

$$(3.9) \quad g^{*ij} = \frac{1}{\tau(1 + 2\rho)}g^{ij} - \frac{1}{\tau^2(1 + 2\rho)}(l^i v^j + v^i l^j) + \frac{\tau(1 + 2\rho) + v^2 - 1}{\tau^3(1 + 2\rho)}l^i l^j$$

in the Finsler space F_n^* .

Differentiating (3.8) partially with respect to y^k , we obtain the Cartan tensor C_{ijk}^* of F_n^* as

$$(3.10) \quad C_{ijk}^* = \tau(1 + 2\rho)C_{ijk} + \frac{(1 + 2\rho)}{2L}(h_{jk}c_i + h_{ki}c_j + h_{ij}c_k).$$

Here $c_i = v_i - (\tau - 1)l_i$. Thus, we have

Theorem 3.1. *The components of the metric tensor g_{ij}^* , the inverse metric tensor g^{*ij} , the angular metric tensor h_{ij}^* and the Cartan tensor C_{ijk}^* of the Finsler space F_n^* whose metric L^* is obtained from the metric L of the Finsler space F_n by (1.1), are given by (3.8), (3.9), (3.7) and (3.10) respectively.*

4. The Cartan connection of the Finsler space $F_n^* = (M_n, L^*)$

In this section, we find the Cartan connection of the Finsler space $F_n^* = (M_n, L^*)$. Since L_{ij} is h -covariant constant with respect to the Cartan connection $CT = (F_{jk}^i, G_j^i, C_{jk}^i)$, i.e. $L_{ij|k} = 0$, (2.1) gives

$$(4.1) \quad \partial_k L_{ij} = L_{ijr}F_{0k}^r + L_{rj}F_{ik}^r + L_{ir}F_{jk}^r,$$

where $L_{ijk} = \partial_k L_{ij}$ and $F_{0k}^r = F_{ik}^r y^i = G_k^r$.

Differentiating (3.5) covariantly with respect to x^i , we get

$$(4.2) \quad \partial_i L_{jk}^* = (1 + 2\rho)\partial_i L_{jk} + 2\rho_i L_{jk},$$

where $\partial_i \rho = \rho_i$.

In view of (4.1), (3.5) and (3.6), (4.2) can be written as

$$(4.3) \quad (1 + 2\rho)\{L_{jkr}(F_{0i}^{*r} - F_{0i}^r) + L_{kr}(F_{ji}^{*r} - F_{ji}^r) + L_{jr}(F_{ki}^{*r} - F_{ki}^r)\} = 2\rho_i L_{jk}.$$

Let D_{jk}^i be the difference of the connections F_{jk}^{*i} and F_{jk}^i , i.e.

$$(4.4) \quad D_{jk}^i = F_{jk}^{*i} - F_{jk}^i.$$

In view of (4.4), (4.3) reduces to

$$(4.5) \quad (1 + 2\rho)(L_{jkr}D_{0i}^r + L_{rk}D_{ji}^r + L_{jr}D_{ki}^r) = 2\rho_i L_{jk}.$$

Cyclic rotation of the indices i, j and k gives

$$(4.6) \quad (1 + 2\rho)(L_{kir}D_{0j}^r + L_{ri}D_{jk}^r + L_{kr}D_{ij}^r) = 2\rho_j L_{ki},$$

and

$$(4.7) \quad (1 + 2\rho)(L_{ijr}D_{0k}^r + L_{rj}D_{ki}^r + L_{ir}D_{jk}^r) = 2\rho_k L_{ij}.$$

Using $L_{k|j} = 0$ in (2.1), we have

$$(4.8) \quad \partial_j L_k = L_{kr} F_{0j}^r + L_r F_{jk}^r.$$

Differentiating (3.2) partially with respect to x^j and then using (4.8) and (2.1), we obtain

$$(4.9) \quad L_{kr}^* F_{0j}^* + L^* F_{kj}^* = (1 + 2\rho) L_{kr} F_{0j}^r + (L_r + v_r) F_{kj}^r + v_{k|j}.$$

In view of (3.2), (3.3), (3.5), and (4.4), (4.9) reduces to

$$(4.10) \quad (1 + 2\rho) L_{kr} D_{0j}^r + (l_r + v_r) D_{kj}^r = v_{k|j}.$$

Here subscript '0' denotes the contraction by the supporting element y^k .

Now, we propose

Theorem 4.1. *If $F_n = (M_n, L)$ and $F_n^* = (M_n, L^*)$ are two Finsler spaces over the same manifold M_n and $L^*(x, y)$ is given by (1.1), then the Cartan connection of F_n^* is completely determined by (4.5) and (4.10).*

To prove Theorem 4.1, first we have to prove the following lemma

Lemma 4.1. *The system of equations*

$$(4.11) \quad \begin{aligned} (a) \quad & (1 + 2\rho) L_{jk} A^k = B_j \\ (b) \quad & (l_k + v_k) A^k = B \end{aligned}$$

has a unique solution

$$(4.12) \quad A^k = (1 + 2\rho)^{-1} L B^k + \tau^{-1} (B - (1 + 2\rho) L B_v) l^k,$$

where $\tau = (L^*/L)$, $B_v = B_i v^i$ and $B^i = g^{ij} B_j$ for given B_j and B such that $B_j l^j = 0$.

Proof. From $h_{jk} = L L_{jk}$ and (2.6), (4.11(a)) can be written as

$$(4.13) \quad g_{jk} A^k = (1 + 2\rho)^{-1} L B_j + l_j (l_k A^k).$$

Transvecting (4.13) with v^j , we get

$$(4.14) \quad v_k A^k = (1 + 2\rho)^{-1} L B_v + (\tau - 1) l_k A^k,$$

where $B_v = B_i v^i$.

In view of (4.11(b)), (4.14) implies

$$(4.15) \quad l_k A^k = \tau^{-1} (B - (1 + 2\rho)^{-1} L B_v).$$

Thus, from (4.13) and (4.15), we have

$$(4.16) \quad g_{jk} A^k = (1 + 2\rho)^{-1} L B_j + \tau^{-1} (B - (1 + 2\rho)^{-1} L B_v) l_j.$$

Contraction of (4.16) by g^{ij} gives the solution

$$(4.17) \quad A^i = (1 + 2\rho)^{-1} L B^i + \tau^{-1} (B - (1 + 2\rho)^{-1} L B_v) l^i$$

of the given system, where $B^i = g^{ij} B_j$ and $\tau = \frac{L^*}{L}$. \square

Thus, we are in a position to prove Theorem 4.1. We complete the proof of Theorem 4.1 if we find the value of D_{jk}^i .

We will find the value of D_{jk}^i in three steps. In the first step, we will find the value of D_{00}^i , in the second step we will find D_{j0}^i and in the last step we will find D_{jk}^i .

In view of (4.10), we have

$$(4.18) \quad (1 + 2\rho)L_{jr}D_{0k}^r + (l_r + v_r)D_{jk}^r = v_{j|k}.$$

Simultaneously adding and subtracting (4.18) and (4.10), we get

$$(4.19) \quad (1 + 2\rho)(L_{jr}D_{0k}^r + L_{kr}D_{0j}^r) + 2(l_r + v_r)D_{jk}^r = v_{j|k} + v_{k|j}$$

and

$$(4.20) \quad (1 + 2\rho)(L_{jr}D_{0k}^r - L_{kr}D_{0j}^r) = v_{j|k} - v_{k|j}.$$

If we take

$$(4.21) \quad (a) \quad v_{j|k} + v_{k|j} = 2s_{jk} \quad (b) \quad v_{j|k} - v_{k|j} = 2t_{jk},$$

then (4.19) and (4.20) become

$$(4.22) \quad (1 + 2\rho)(L_{jr}D_{0k}^r + L_{kr}D_{0j}^r) + 2(l_r + v_r)D_{jk}^r = 2s_{jk}.$$

$$(4.23) \quad (1 + 2\rho)(L_{jr}D_{0k}^r - L_{kr}D_{0j}^r) = 2t_{jk}.$$

Subtracting (4.7) from the addition of (4.5) and (4.6), we have

$$(4.24) \quad 2L_{kr}D_{ij}^r + (1 + 2\rho)(L_{jkr}D_{0i}^r + L_{kir}D_{0j}^r - L_{ijr}D_{0k}^r) = 2(\rho_i L_{jk} + \rho_j L_{ki} - \rho_k L_{ij}).$$

Transvection of (4.22), (4.23) and (4.24) with y^k and utilization of $L_{ij}y^j = 0$ give us

$$(4.25) \quad (1 + 2\rho)L_{jr}D_{00}^r + 2(l_r + v_r)D_{0j}^r = 2s_{j0},$$

$$(4.26) \quad (1 + 2\rho)L_{jr}D_{00}^r = 2t_{j0},$$

$$(4.27) \quad (1 + 2\rho)(L_{jir}D_{0i}^r + L_{ir}D_{0j}^r + L_{ijr}D_{00}^r) = 2\rho_0 L_{ij}.$$

Transvecting (4.25) with y^j , we find

$$(4.28) \quad (l_r + v_r)D_{00}^r = s_{00}.$$

Applying Lemma 4.1 in (4.26) and (4.28), we get

$$(4.29) \quad D_{00}^r = \frac{L}{L^*(1 + 2\rho)} \{2L^*t_0^r + l^r((1 + 2\rho)s_{00} - 2Lt_{v0})\}.$$

Here $t_0^r = g^{ir} t_{i0}$ and $t_{v0} = t_{i0} v^i$.

Putting k in place of i in (4.27) and then adding with (4.23), we find

$$(4.30) \quad L_{jr} D_{0k}^r = \frac{1}{2(1+2\rho)} (2t_{jk} + 2\rho_0 L_{jk} - \frac{1}{2}(1+2\rho) L_{jkr} D_{00}^r).$$

If we take

$$(4.31) \quad \frac{1}{2(1+2\rho)} (2t_{jk} + 2\rho_0 L_{jk} - \frac{1}{2}(1+2\rho) L_{jkr} D_{00}^r) = A_{jk},$$

then (4.30) reduces to

$$(4.32) \quad L_{jr} D_{0k}^r = A_{jk}.$$

From (4.29) and (4.31), we have

$$(4.33) \quad A_{jk} = \frac{1}{2L^*(1+2\rho)} \{2L^*(t_{jk} - LL_{jkr} t_0^r) + L_{jk}((1+2\rho)s_{00} - 2Lt_{v0} + 2L^*\rho_0)\}.$$

This shows that A_{jk} is known.

If we write

$$(4.34) \quad s_{k0} - \frac{1}{2}(1+2\rho) L_{kr} D_{00}^r = A_k,$$

the equation (4.25) assumes the form

$$(4.35) \quad (l_r + v_r) D_{0k}^r = A_k.$$

Putting the value of D_{00}^r from (4.29) in (4.34), we get

$$(4.36) \quad A_k = s_{k0} - LL_{kr} t_0^r.$$

In view of Lemma 4.1, the system of equations (4.32) and (4.35) give

$$(4.37) \quad D_{0k}^r = \frac{L}{L^*} (L^* A_k^r + l^r (A_k - LA_{vk})),$$

where $A_{vk} = A_{jk} v^j$ and $A_k^r = g^{ri} A_{ik}$.

Now we can express (4.22) in the form

$$(4.38) \quad (l_r + v_r) D_{jk}^r = B_{jk},$$

where

$$(4.39) \quad B_{jk} = s_{jk} - \frac{1}{2}(1+2\rho) (L_{jr} D_{0k}^r + L_{kr} D_{0j}^r).$$

The equation (4.24) may be written as

$$(4.40) \quad L_{ir} D_{jk}^r = B_{ijk},$$

where

$$(4.41) \quad B_{ijk} = (\rho_j L_{ki} + \rho_k L_{ij} - \rho_i L_{jk}) - \frac{1}{2}(1 + 2\rho)(L_{kir} D_{0j}^r + L_{ijr} D_{0k}^r - L_{jkr} D_{0i}^r).$$

Putting the value of D_{0i}^r from (4.37), we see that B_{jk} and B_{ijk} are known quantities. Applying the Lemma 4.1, for the system of equations (4.38) and (4.40), we obtain

$$(4.42) \quad D_{jk}^r = \frac{L}{L^*} \{L^* B_{jk}^r + l^r (B_{jk} - L B_{vjk})\},$$

where $B_{jk}^r = g^{ir} B_{ijk}$ and $B_{vjk} = B_{ijk} v^i$. The quantities B_{jk} and B_{ijk} are given by respectively (4.39) and (4.41) together with (4.37).

Thus, the proof is completed.

5. Subspace of the Finsler space $F_n^* = (M_n, L^*)$

Suppose F_m and F_m^* are the subspaces of the Finsler spaces F_n and F_n^* respectively.

Contracting (2.14) by w^α and using (2.9) and $y^i g_{ij} = y_j$, we obtain

$$(5.1) \quad y_j Y_{(a)}^j = 0.$$

Again contracting (3.8) with $Y_{(a)}^i Y_{(b)}^j$ and using (2.16), (5.1) and $\tau = (L^*/L)$, we have

$$(5.2) \quad g_{ij}^* Y_{(a)}^i Y_{(b)}^j = \frac{L^*}{L} (1 + 2\rho) \delta_{(a)(b)} + v_i Y_{(a)}^i v_k Y_{(b)}^k.$$

Fixing the index (a) and taking $(a) = (b)$ in (5.2), we get

$$(5.3) \quad g_{ij}^* Y_{(a)}^i Y_{(a)}^j = \frac{L^*}{L} (1 + 2\rho) + (v_r Y_{(a)}^r)^2.$$

Hence

$$(5.4) \quad g_{ij}^* \left(\frac{Y_{(a)}^i}{\sqrt{\frac{L^*}{L} (1 + 2\rho) + (v_r Y_{(a)}^r)^2}} \right) \left(\frac{Y_{(a)}^j}{\sqrt{\frac{L^*}{L} (1 + 2\rho) + (v_r Y_{(a)}^r)^2}} \right) = 1.$$

From (5.4), it is clear that $\left(\frac{Y_{(a)}^i}{\sqrt{\frac{L^*}{L} (1 + 2\rho) + (v_r Y_{(a)}^r)^2}} \right)$ is a unit vector.

Contracting (3.8) by $B_\alpha^i Y_{(a)}^j$ and using (2.14) & (5.1), we obtain

$$(5.5) \quad g_{ij}^* B_\alpha^i Y_{(a)}^j = (v_j Y_{(a)}^j) (v_i + l_i) B_\alpha^i.$$

From (5.5), we can say that $Y_{(a)}^j$ is normal to the subspace F_m^* if and only if the condition

$$(5.6) \quad (v_j Y_{(a)}^j) (v_i + l_i) B_\alpha^i = 0$$

holds. This implies at least one of the conditions $v_j Y_{(a)}^j = 0$ and $(v_i + l_i) B_\alpha^i = 0$. Suppose $(v_i + l_i) B_\alpha^i = 0$. Contracting this condition by w^α and using $B_\alpha^i w^\alpha = y^i$, we get $L + v_i y^i = L^* = 0$ which is not possible. Hence, we have the first condition, i.e.

$$(5.7) \quad v_j Y_{(a)}^j = 0.$$

Thus, the vector $Y_{(a)}^j$ is normal to the subspace F_m^* if and only if the vector v_j is tangent to the subspace F_m . From (5.4), (5.5) and (5.7), we find that $\left(\frac{Y_{(a)}^j}{\sqrt{\frac{L^*}{L}(1+2\rho)}} \right)$ is a unit normal vector of the subspace F_m^* . In view of (2.14), (2.15) and (2.16), we obtain

$$(5.8) \quad Y_{(a)}^{*j} = \frac{Y_{(a)}^j}{\sqrt{\frac{L^*}{L}(1+2\rho)}}.$$

Contracting (3.8) by $Y_{(a)}^{*i}$ and using (2.15), we obtain

$$(5.9) \quad Y_j^{*(a)} = \sqrt{\frac{L^*}{L}(1+2\rho)} Y_i^{(a)}.$$

Thus, we have

Theorem 5.1. *Let $F_n^* = (M_n, L^*)$ be a Finsler space whose metric function L^* is obtained from the metric function L of the Finsler space $F_n = (M_n, L)$ by the transformation (1.1). If F_m^* and F_m are m -dimensional subspaces of F_n^* and F_n respectively, then the vector $v_i(x, y)$ satisfying the condition (1.2) is tangent to F_m if and only if any vector $Y_{(a)}^i$ normal to F_m is also normal to F_m^* .*

Let us assume that the transformation given by (1.1) is projective. Then, we have

$$(5.10) \quad G_j^{*i} = G_j^i + p_j y^i + p \delta_j^i,$$

where p is a function of directional argument and of homogeneity one in y^i and $\partial_j p = p_j$.

Contracting (4.4) by y^k and using $F_{jk}^i y^k = G_j^i$, we get

$$(5.11) \quad G_j^{*i} = G_j^i + D_{0j}^i.$$

Thus, from (5.10) and (5.11), we have

$$(5.12) \quad D_{0j}^i = p_j y^i + p \delta_j^i.$$

Contracting (5.12) by $B_\alpha^j Y_i^{(a)}$, using (2.15) and (2.19(a)), we obtain

$$(5.13) \quad Y_i^{(a)} D_{0j}^i B_\alpha^j = 0.$$

If every geodesic in the subspace F_m with respect to the induced metric is also a geodesic in the enveloping space F_n , the subspace F_m is called totally geodesic subspace and this type of space is characterized by

$$(5.14) \quad H_\alpha^{(a)} = 0,$$

i.e. its normal curvature vector vanishes identically.

In view of (2.21), the normal curvature vector $H_\alpha^{*(a)}$ of the subspace F_m^* in the direction $Y_i^{(a)}$ is given by

$$(5.15) \quad H_\alpha^{*(a)} = Y_i^{*(a)} (B_{0\alpha}^i + G_j^{*i} B_\alpha^j).$$

Using (5.9) and (5.11) in (5.15), we get

$$(5.16) \quad H_\alpha^{*(a)} = \sqrt{\frac{L^*}{L}(1+2\rho)} H_\alpha^{(a)} + \sqrt{\frac{L^*}{L}(1+2\rho)} Y_i^{(a)} D_{0j}^i B_\alpha^j = 0.$$

In view of (5.13), (5.16) reduces to

$$(5.17) \quad H_\alpha^{*(a)} = \sqrt{\frac{L^*}{L}(1+2\rho)} H_\alpha^{(a)}.$$

$\sqrt{\frac{L^*}{L}(1+2\rho)} \neq 0$ for $\sqrt{\frac{L^*}{L}(1+2\rho)} = 0$ implies $\rho = -(1/2)$, a contradiction to the fact that ρ is a function of x^i . Hence, we conclude from (5.17) that $H_\alpha^{(a)}$ vanishes if and only if $H_\alpha^{*(a)}$ vanishes as $\sqrt{\frac{L^*}{L}(1+2\rho)} \neq 0$. Therefore, we have

Theorem 5.2. *If a contravariant vector field v^i satisfying the condition (1.2) is tangent to a subspace F_m of the space F_n then F_m is totally geodesic if and only if the subspace F_m^* of F_n^* is totally geodesic.*

If there exists a projective change between the Finsler spaces $F_n = (M_n, L)$ and $F_n^* = (M_n, L^*)$ over the underlying manifold M_n such that the later space is locally Minkowskian then the space F_n is said to be projectively flat.

In 2005, M. Kitayama [7] showed that a totally geodesic subspace of a projectively flat Finsler space is also projective.

Makato Matsumoto [8] proved that a Finsler space $F^n (n > 3)$ is projectively flat if the Weyl torsion tensor W_{jk}^i and the Douglas tensor D_{jkh}^i vanish, i.e.

$$(5.18) \quad (a) \quad W_{jk}^i = 0, \quad (b) \quad D_{jkh}^i = 0$$

and the converse part is also true.

Under a projective change $W_{jk}^{*i} = W_{jk}^i$ and $D_{jkh}^{*i} = D_{jkh}^i$, i.e. both tensors are invariant [10]. Thus, we conclude following theorem, from Theorem 5.2 in view of (5.18)

Theorem 5.3. *Let the metric function L^* of a Finsler space $F_n^* = (M_n, L^*)$ be obtained from the metric function L of a projective flat Finsler space $F_n = (M_n, L), n > 3$, by the transformation (1.1). If a subspace F_m of F_n is totally geodesic and the vector field v^i satisfying (1.2) is tangential to it, then the corresponding subspace F_m^* of F_n^* is projectively flat.*

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